HW 2 (today): 2.2 & slope fields
HW 3 (Wed): 2.1 Integrating Factors
2.3 applications

2.1: Integrating Factors (Linear 1st Order Diff. Eqns)

Recall: We have been solving 1st order equations such as:

$$\frac{dy}{dt} = G(t,y)$$

We call the equation ***linear*** if $G(t,y)$ is a linear function of *y*, that is:

$$G(t,y) = m(t)y + b(t)$$

y^1 ← FIRST POWER ONLY

Otherwise, we say it is **non-linear.**

Many important applications we have seen are linear (populations, bank accounts, air resistance, temperature, etc…), so this is an important special case.

Given a 1st order linear ODE, we like to re-arrange it into the form:

$$\frac{dy}{dt} + p(t)y = g(t)$$

Examples:

1. $\frac{dy}{dt} + ty = t^3$

Give $p(t)$ and $g(t)$.

$p(t) = t$

$g(t) = t^3$

2. $x\frac{dy}{dx} = \sin(x) - 2y$

Give $p(x)$ and $g(x)$.

$x\frac{dy}{dx} + 2y = \sin(x)$

$\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin(x)}{x}$

$p(x) = \frac{2}{x}$, $g(x) = \frac{\sin(x)}{x}$

Big Observation 1

The form

$$\frac{dy}{dt} + p(t)y$$

looks sort of like the *product rule.*

Recall: Here is the product rule

$$\frac{d}{dt}(f(t)y) = f(t)\frac{dy}{dt} + f'(t)y$$

Example:

$$\frac{dy}{dt} + 2y = \frac{t}{e^{2t}}$$

so $p(t) = 2$ and $g(t) = \frac{t}{e^{2t}}$

What happens if you multiply both sides by e^{2t}?

$$\underbrace{e^{2t}\frac{dy}{dt} + 2e^{2t}y} = t$$

$$\frac{d}{dt}(e^{2t}y) = t$$

$$\int \frac{d}{dt}(e^{2t}y)\,dt = \int t\,dt$$

$$e^{2t}y = \frac{1}{2}t^2 + c$$

$$\boxed{y = \frac{\frac{1}{2}t^2 + c}{e^{2t}}}$$

You get

$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = t$$

which is

$$\frac{d}{dt}(e^{2t}y) = t$$

Integrating both sides gives

$$e^{2t}y = \frac{1}{2}t^2 + C$$

so

$$y = \frac{\frac{1}{2}t^2 + C}{e^{2t}}$$

In this example, we call $\mu(t) = e^{2t}$, the *integrating factor,* which is a function we multiply by so we can reverse the product rule.

Okay, sort of cool, but we were lucky this time, how can we make this work in a more general way?

We need another big observation.

Big Observation 2

If $F(t)$ is any antiderivative of $p(t)$

$$F(t) = \int p(t)dt$$

then

$$\frac{d}{dt}\left(e^{F(t)}y\right) = e^{F(t)}\frac{dy}{dt} + p(t)e^{F(t)}y$$
$$= e^{F(t)}\left(\frac{dy}{dt} + p(t)y\right).$$

So if we multiply by

$$\mu(t) = e^{\int p(t)dt}$$

then we create a situation where we can reverse the product rule.

$$\underbrace{\mu(t)\frac{dy}{dt} + \mu(t)p(t)y} = \mu(t)g(t)$$

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(t)\,g(t)$$

$$\boxed{y = \frac{1}{\mu(t)}\int \mu(t)g(t)dt}$$

ALWAYS "WORKS"

Integrating Factor Method

Given a linear, 1st order ODE

$$\frac{dy}{dt} = f(t, y)$$

Step 0: Put in form

$$\frac{dy}{dt} + p(t)y = g(t)$$

Step 1: Find $F(t) = \int p(t)dt$

& write/simplify $\mu(t) = e^{\int p(t)dt}$

Step 2: Multiply BOTH sides by $\mu(t)$.

& re-write LHS as product rule.

Step 3: Integrate both sides, and simplify.

Example: $\frac{dy}{dt} = \frac{\cos(t)}{t^2} - \frac{2y}{t}$

Step 0:

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{\cos(t)}{t^2}$$

Step 1: $F(t) = \int \frac{2}{t}dt = 2\ln(t) + C$

$$\mu(t) = e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

Step 2: $t^2\frac{dy}{dt} + 2ty = \cos(t)$

$$\frac{d}{dt}(t^2y) = \cos(t)$$

Step 3: $t^2y = \sin(t) + C$

$$y = \frac{\sin(t) + C}{t^2}$$

Example:

$$3\frac{dy}{dt} - 6ty - 3e^{t^2} = 0$$

$$\left.\begin{aligned} \frac{dy}{dt} - 2ty - e^{t^2} &= 0 \\ \frac{dy}{dt} - 2ty &= e^{t^2} \end{aligned}\right\} \quad \begin{aligned} p(t) &= -2t \\ g(t) &= e^{t^2} \end{aligned}$$

$\boxed{\text{I}}$ $\boxed{\text{Integrating Factor}}$

$$\int p(t)\,dt = \int -2t\,dt = -t^2 + c_0$$

$$\mu(t) = e^{-t^2}$$

$\boxed{\text{II}}$ $\boxed{\text{multiply}}$

$$e^{-t^2}\frac{dy}{dt} - 2te^{-t^2}y = \overbrace{e^{-t^2}e^{t^2}}^{e^0 = 1}$$

$$\frac{d}{dt}\left(e^{-t^2}y\right) = 1$$

$\boxed{\text{III}}$ $\boxed{\text{INTEGRATE}}$

$$e^{-t^2}y = t + C$$

$$\boxed{y = (t+c)e^{t^2}}$$

Example:

$$\frac{dy}{dt} = t - 3y$$

$$\frac{dy}{dt} + 3y = t$$

$p(t) = 3, \quad g(t) = t$

$\boxed{\text{I}}$ $\mu(t) = e^{\int 3\,dt} = e^{3t}$

$\boxed{\text{II}}$ $e^{3t}\frac{dy}{dt} + 3e^{3t}y = te^{3t}$

$$\frac{d}{dt}\left(e^{3t}y\right) = te^{3t}$$

$\boxed{\text{III}}$

$$e^{3t}y = \int te^{3t}\,dt$$

$$e^{3t}y = \frac{1}{3}te^{3t} - \int \frac{1}{3}e^{3t}\,dt$$

$u = t \qquad dv = e^{3t}dt$

$du = dt \qquad v = \frac{1}{3}e^{3t}$

$$e^{3t}y = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C$$

$$\boxed{y = \frac{1}{3}t - \frac{1}{9} + Ce^{-3t}}$$

Two Notes:

- Only for linear 1st order ODEs!

  Aside:

  Again, sometimes a substitution can make it linear.

  Example:

  $$e^y \frac{dy}{dx} - \frac{1}{x} e^y = 3x \quad \text{is not linear}$$

  Using

  $$v = e^y \rightarrow \frac{dv}{dt} = e^y \frac{dy}{dt}$$

  Changes it to

  $$\frac{dv}{dx} - \frac{1}{x} v = 3x \quad \text{which is linear.}$$

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = \frac{1}{x}$$

$$\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2} v = 3$$

$$\frac{d}{dx}\left(\frac{1}{x} v\right) = 3 \Rightarrow \frac{1}{x} v = 3x + C \quad \boxed{v = 3x^2 + Cx} \quad v = e^y \quad \boxed{y = \ln(3x^2 + Cx)}$$

- If we can't do the integrals we can still write our answer in terms of integrals.

In which case the convention is to write

$$\int f(t)dt = \int_0^t f(u)du + C$$

(so we can solve for C if needed).

See next page for an example.

Example:

$$\frac{1}{6}\frac{dy}{dt} + t^2 y = \frac{1}{6}$$

$\frac{dy}{dt} + 6t^2 y = 1$

$p(t) = 6t^2, \quad g(t) = 1$

$\boxed{\text{I}}$ $\mu(t) = e^{2t^3}$

$\boxed{\text{II}}$ $e^{2t^3}\frac{dy}{dt} + 6t^2 e^{2t^3} y = e^{2t^3}$

$\frac{d}{dt}\left(e^{2t^3} y\right) = e^{2t^3}$

$\boxed{\text{III}}$

$e^{2t^3} y = \underbrace{\int e^{2t^3}\,dt}_{\text{CAN'T DO}}$

$e^{2t^3} y = \int_0^t e^{2u^3}\,du + C$

$$\boxed{y = e^{-2t^3}\int_0^t e^{2u^3}\,du + Ce^{-2t^3}}$$

THE GENERAL SOL'N TO $\frac{dy}{dt} + p(t)y = g(t)$ IS

$$y = e^{-\int p(t)dt}\int g(t)\,e^{\int p(t)dt}\,dt$$